Selected Readings in Micro: Multi-Echelon System

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Multi-Echelon Production/Inventory Theory

- these lecture notes are based on G.J. van Houtum's review paper: "Multi-Echelon Production/Inventory Systems: Optimal Policies, Heuristics, and Algorithms," appeared in *Tutorials in Operations Research*, 2006 INFORMS

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- multi-echelon theory: originated from Clark and Scarf ("Optimal policies for a multi-echelon inventory problem," *Management Science*, 1960)
- Clark-Scarf model: a supply chain consisting of multiple stages with a serial structure
  - stage N orders at an external supplier, stage $N-1$ orders at stage N, ..., stage 1 faces external demand
  - stage may represent a production node, or a transportation node
  - time consists of periods of equal length, e.g., days, weeks or months, and the time horizon is infinite

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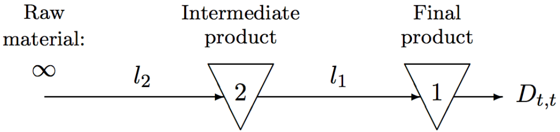
Figure 1: The serial, two-echelon production/inventory system.



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  - an amount ordered by stage 2 (1) arrives at stockpoint 2 (1) after a deterministic leadtime l_2 (l_1)
- inventory constraints
  - there is always sufficient raw material available and thus orders by stockpoint 2 are never delayed
  - if the available stock at stockpoint 2 is smaller than the ordered amount by stage 1, then the available amount is sent to stage 1 while the rest is delivered as soon as possible

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- for simplicity, we assume the demands in different periods are i.i.d., following some distribution F with mean $\mu > 0$

Timing of Events

- in each period, the following events happen in order
  1. at each stage, an order is placed
  2. arrival of orders
  3. demand occurs
  4. one-period costs are assessed

Echelon Inventory Level/Position

- *echelon inventory level* (or *echelon stock*) of a given stockpoint: all physical stock at that stockpoint plus all materials in transit to or on hand at any stockpoint downstream minus eventual backlogs at the most downstream stock points
  - the chain under consideration is called the *echelon*

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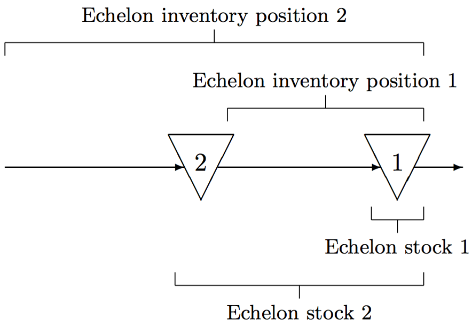
- echelon inventory position of a stockpoint: its echelon inventory level plus all materials which are in transit to the stockpoint

Echelon Inventory Level/Position

Figure 2: The concepts echelon stock and echelon inventory position.



Echelon Cost

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- assumption: the cost functions $c_n(x_n)$, $n = 1, 2$, are convex and coercive (it is suboptimal to let the backlog grow to infinity)
- special case: linear holding and penalty costs

$$\underbrace{h_2}_{\text{holding cost at 1}} (x_2 - x_1) + \underbrace{(h_1 + h_2)}_{\text{holding cost at 2}} (x_1)_+ + \underbrace{p}_{\text{penalty cost}} (x_1)_- = c_2(x_2) + c_1(x_1) \tag{1}$$

where $c_2(x_2) = h_2 x_2$ and $c_1(x_1) = h_1 x_1 + (p + h_1 + h_2)(x_1)_-$

Objective Function

- minimize the average costs per period

$$\min_{\pi \in \Pi} G(\pi) = \lim_{T \to \infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} \left(C_{t,2} + C_{t,1}\right)\right]$$

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- simplification assumptions: no discounting, no ordering cost
  - make the key intuition transparent
  - many insights extend to more complicated settings

Decision Cycle at Time t_0

- stage 2's decision at the beginning of period t_0: order such that the echelon inventory position $IP_{t_0,2}$ becomes some level z_2, which affects the expected cost at the end of period $t_0 + l_2$, i.e.,

$$E\left[\left. C_{t_0+l_2,2} \right| IP_{t_0,2} = z_2\right] = E\left[c_2\left(z_2 - D_{t_0,t_0+l_2}\right)\right]$$

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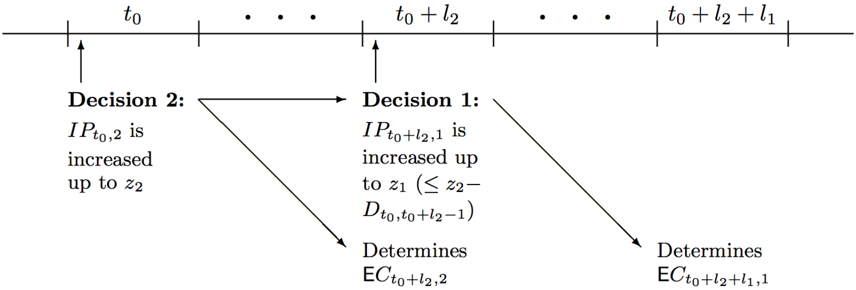
- these two decisions are interdependent: stage 2's decision at $t_0 + l_2$ leads to the echelon inventory level $IL_{t_0+l_2,2} = z_2 - D_{t_0,t_0+l_2-1}$, which directly limits the level to which stage 1 can increase $IP_{t_0+l_2,1}$, i.e.,

$$z_1 \leq z_2 - D_{t_0,t_0+l_2-1}$$

Illustration

Figure 3: The consequences of the decisions 1 and 2.



The Relaxed Single-Cycle Problem

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- this problem has the same structure as a 2-period dynamic programming problem: Clark-Scarf's key insight is to solve the multi-echelon problem by mimicking the idea of dynamic programming, i.e., treating the series of decisions on the supply chain as the series of decision over time

The Optimal Choice for z_1

- define a function

$$G_1(y_1) \equiv E\left[c_1\left(z_1 - D_{t_0+l_2,t_0+l_2+l_1}\right)\right]$$

  - the expected costs attached to echelon 1 at the end of a period $t_0 + l_2 + l_1$ if echelon inventory position 1 at the beginning of period $t_0 + l_2$ has been increased up to level y_1
  - interpretation: cost associated with the subsystem that consists of stage 1 only and has stockpoint 2 as external supplier with infinite supply

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- result: $G_1(y_1)$ is convex in y_1 and $S_1 \equiv \arg\min_{y_1} G_1(y_1)$ defines a order-up-to/base-stock policy

$$\begin{aligned} z_1 &= \min\left\{IL_{t_0+l_2,2}, S_1\right\} \\ &= \min\left\{y_2 - D_{t_0,t_0+l_2-1}, S_1\right\} \end{aligned}$$

  - optimal policy depends on y_2 because of the natural constraint induced by the inventory level at stockpoint 2

Optimal Choice for z_2

- define a function

$$G_2(y_1, y_2) = E\left[c_2\left(y_2 - D_{t_0, t_0+l_2}\right) + c_1\left(\min\left\{y_2 - D_{t_0, t_0+l_2-1}, y_1\right\} - D_{t_0+l_2, t_0+l_2+l_1}\right)\right]$$

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- decomposition result: optimal base-stock levels can be obtained sequentially by the minimization of one-dimensional functions

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where $B_0^{(1)} = \left(D_{t_0+l_2, t_0+l_2+l_1} - S_1\right)_+$ represents the backlog at stockpoint 1 for echelon 1

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- for a given optimal basestock level S_1 for echelon 1, the optimal basestock level S_2 for echelon 2 is such that

$$\Pr\left\{B_0^{(2)} = 0\right\} = \frac{p}{p + h_1 + h_2}$$

  where $B_0^{(2)} = \left(B_1^{(2)} + D_{t_0+l_2,t_0+l_2+l_1} - S_1\right)_+$ (with $B_1^{(2)} = (D_{t_0,t_0+l_2-1} - (S_2 - S_1))_+$) represent the backlog at stockpoint 1 for echelon 2
  - non-stockout probability is lower when both stockpoints are considered

Extensions

- continuous review: continuous-time version of the periodic review version we have discussed
- discrete product: order amounts and inventory levels are discrete variables
- discounted costs: minimizing discounted expected costs
- target service level: minimizing costs subject to service level constraint
- assembly systems, distribution and general systems
- many more ...

Next Class Paper

- Jovanovic, B: "Selection and the evolution of industry," *Econometrica*, 1982